Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Tuesday 23 June 2020

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **9FM0/4A**

Further Mathematics
Advanced
Paper 4A: Further Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B)
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information
- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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P 6 2 6 8 1 A 0 1 3 2

Pearson

PMT

1. A small sports club has 12 adult members and 14 junior members.

The club needs to enter a team of 8 players for a particular competition.

Determine the number of ways in which the team can be selected if

(i) there are no restrictions on the team,
(1)

(ii) the team must contain 4 adults and 4 juniors,
(2)

(iii) more than half the team must be adults.
(3)

(i) $12 + 14 = 26$ $\qquad {}^nC_r = \frac{n!}{(n-r)!r!}$

${}^{26}C_8 = 1562275$ 1

(ii) ${}^{12}C_4 \times {}^{14}C_4$ 1 $= 495 \times 1001 = 495495$ 1

(iii) $8 \div 2 = 4$ so consider cases:

5 adults . ${}^{12}C_5 \times {}^{14}C_3 = 792 \times 364 = 288\,288$

6 adults 1 . ${}^{12}C_6 \times {}^{14}C_2 = 924 \times 91 = 84\,084$

7 adults . ${}^{12}C_7 \times {}^{14}C_1 = 792 \times 14 = 11\,088$

8 adults . ${}^{12}C_8 \times {}^{14}C_0 = 495 \times 1 = 495$ 1

$288\,288 + 84\,084 + 11\,088 + 495 = 383\,955$ 1

Remember! + for 'OR', × for 'AND'.

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P 6 2 6 8 1 A 0 2 3 2

Question 1 continued

(Total for Question 1 is 6 marks)

2. Solve the recurrence system

$$u_1 = 1 \qquad u_2 = 4$$
$$9u_{n+2} - 12u_{n+1} + 4u_n = 3n$$

(9)

Find auxiliary equation.

$9u_{n+2} - 12u_{n+1} + 4u_n = 0$

$9r^2 - 12r + 4 = 0$ ① solve for r

$(3r-2)^2 = 0$

$r = \frac{2}{3}$ ① is a repeated root, so the complementary function has the form $CF = (A + Bn) \times \left(\frac{2}{3}\right)^n$ ①

$u_n = CF + PS$ so find PS:

$3n$ has form $pn+q$ so PS has form $an+b$

Substitute into original:

$9(a(n+2)+b) - 12(a(n+1)+b) + 4(an+b) = 3n$ ①

$9(an+2a+b) - 12(an+a+b) + 4(an+b) = 3n$

$(9-12+4)an + (9\times2 - 12\times1)a + (9-12+4)b = 3n$

$an + 6a + b = 3n \quad \therefore a = 3$ ①

$6a + b = 0 \Rightarrow 18 + b = 0 \Rightarrow b = -18$ ①

$u_n = (A + Bn) \times \left(\frac{2}{3}\right)^n + 3n - 18$ ①

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Question 2 continued

Use $u_1 = 1$ and $u_2 = 4$ to find A, B:

$1 = (A + B(1)) \times \left(\frac{2}{3}\right)^1 + 3(1) - 18$

$1 = \frac{2}{3}(A + B) - 15$

$16 \times \frac{3}{2} = A + B$

① $24 = A + B$

$4 = \left(\frac{2}{3}\right)^2 \times (A + B(2)) + 3(2) - 18$

$4 = \frac{4}{9}(A + 2B) - 12$

$16 \times \frac{9}{4} = A + 2B$

② $36 = A + 2B$

② − ① : $B = 12$

① $A + 12 = 24 \Rightarrow A = 12$ ①

$\therefore u_n = 12(n+1)\left(\frac{2}{3}\right)^n + 3n - 18$ ①

Question 2 continued

Question 2 continued

(Total for Question 2 is 9 marks)

3.

$$\mathbf{M} = \begin{pmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

where k is a constant.

(a) Show that, in terms of k, a characteristic equation for $\mathbf{M}$ is given by

$$\lambda^3 - (2k + 13)\lambda + 5(k + 6) = 0$$

(3)

Given that $\det \mathbf{M} = 5$

(b) (i) find the value of k

(ii) use the Cayley-Hamilton theorem to find the inverse of $\mathbf{M}$.

(7)

(a) $\det(A - \lambda I) = 0$

$$\left| \begin{bmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & k & -2 \\ 2 & -4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{vmatrix} \quad 1 \quad = (1-\lambda)[(-4-\lambda)(3-\lambda) - (1\times 2)]$$

$$- k[2(3-\lambda) - 1\times 1] - 2[2\times 2 - (-4-\lambda)] \quad 1$$

$$0 = (1-\lambda)(-14 + \lambda + \lambda^2) - k(5 - 2\lambda) - 2(8 + \lambda)$$

$$0 = -14 + \lambda + \lambda^2 + 14\lambda - \lambda^2 - \lambda^3 - 5k + 2k\lambda - 16 - 2\lambda$$

$$0 = -\lambda^3 + (2k + 13)\lambda - 5(k + 6)$$

$$0 = \lambda^3 - (2k + 13)\lambda + 5(k + 6) \quad 1$$

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Question 3 continued

(b)(i) $\lambda^3 - (2k+13)\lambda \pm 5(k+6) = 5$

$5(k+6) = 5$ ①

$k+6 = 1$

$k = -7$ ①

Cayley-Hamilton Theorem

Every square matrix is a solution to it's own characteristic equation.

(ii) Hence by the C-H theorem.

$$M^3 + M - 5I = 0 \quad ①$$

$$\times M^{-1}$$

$$M^2 + I - 5M^{-1} = 0 \quad ①$$

$$5M^{-1} = M^2 + I$$

$$\div 5$$

$$M^{-1} = \frac{1}{5}(M^2 + I) \quad ①$$

$$M^2 = \begin{pmatrix} 1 & -7 & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -7 & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$M^2 + I = \begin{pmatrix} -15 & 17 & -15 \\ -5 & 4 & -5 \\ 8 & -9 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ①$$

$$M^{-1} = \frac{1}{5}\begin{pmatrix} -14 & 17 & -15 \\ -5 & 5 & -5 \\ 8 & -9 & 10 \end{pmatrix} \quad ①$$

Question 3 continued

Question 3 continued

(Total for Question 3 is 10 marks)

4.

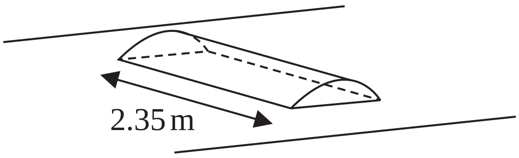

Figure 1

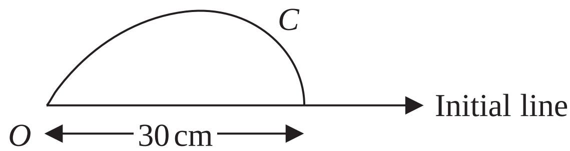

Figure 2

Figure 1 shows a sketch of a design for a road speed bump of width 2.35 metres. The speed bump has a uniform cross-section with vertical ends and its length is 30 cm. A side profile of the speed bump is shown in Figure 2.

The curve C shown in Figure 2 is modelled by the polar equation

$$r = 30(1 - \theta^2) \qquad 0 \leqslant \theta \leqslant 1$$

The units for r are centimetres and the initial line lies along the road surface, which is assumed to be horizontal.

Once the speed bump has been fixed to the road, the visible surfaces of the speed bump are to be painted.

Determine, in cm^2, the area that is to be painted, according to the model.

(10)

Use integration to find the area under the curve C.

$$\text{Area} = \int \frac{1}{2} r^2 \, d\theta$$

$$= \int \frac{1}{2} \left[30(1-\theta^2)\right]^2 d\theta \quad 1$$

$$= \int \frac{900}{2} (1-\theta^2)^2 \, d\theta$$

$$= 450 \int_0^1 \theta^4 - 2\theta^2 + 1 \, d\theta$$

$$= 450 \left[\theta - \frac{2}{3}\theta^3 + \frac{1}{5}\theta^5\right]_0^1 \quad 1$$

$$= 450 \left[1 - \frac{2}{3} + \frac{1}{5}\right]$$

Question 4 continued

$= 240\text{ cm}^2$ ①

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left[30(1-\theta^2)\right]^2 + \left[30(-2\theta)\right]^2$ ①

$= 900(1-2\theta^2+\theta^4) + 3600\theta^2$

$= 900\theta^4 + 1800\theta^2 + 900$

$= 900(1+\theta^2)^2$ ①

Length of curve is:

$\int_0^1 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 30\int_0^1 1+\theta^2\, d\theta$

$= 30\left[\theta + \frac{1}{3}\theta^3\right]_0^1$ ①

$= 30\left(1 + \frac{1}{3} - 0\right)$

$= 40\text{ cm}$ ①

Total surface area:

$2 \times 240 + 235 \times 40 = 9880\text{ cm}^2$ ① ①

① for correct method.

Question 4 continued

Question 4 continued

(Total for Question 4 is 10 marks)

5. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{1 - 3z}{z + 2\mathrm{i}} \qquad z \neq -2\mathrm{i}$$

The circle with equation $|z + \mathrm{i}| = 3$ is mapped by T onto the circle C.

(a) Show that the equation for C can be written as

$$3|w + 3| = |1 + (3 - w)\mathrm{i}|$$

(4)

(b) Hence find

(i) a Cartesian equation for C,

(ii) the centre and radius of C.

(6)

(a) $w = \dfrac{1-3z}{z+2i}$ rearrange for $z = \ldots$

$(z+2i)w = 1-3z$ 1

$wz + 2iw = 1-3z$

$wz + 3z = 1-2iw$

$(3+w)z = 1-2iw$

$z = \dfrac{1-2iw}{3+w}$ 1

substitute $z = .$ into equation of circle to get circle in terms of w

$|z+i| = 3$

$\left|\dfrac{1-2iw}{3+w} + i\right| = 3$

$|1-2iw+(3+w)i| = 3|3+w|$ 1

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Question 5 continued

$|1+3i-2i\omega+\omega| = 3|3+\omega|$

$|1+(3-\omega)i| = 3|3+\omega|$ ①

(b)(i) $\omega = u+iv$ ← since ω is a complex number, it has real (u) and imaginary (v) parts.

$|1+(3-u-vi)i| = 3|u+3+vi|$ ①

$|1+3i-ui+v| = 3|u+3+vi|$

$|(3-u)i+(v+1)| = 3|u+3+vi|$

split into real and imaginary parts.

$(3-u)^2+(v+1)^2 = 9[(u+3)^2+v^2]$ ②

square everything to remove modulus ($|x|$).

(ii) $9-6u+u^2+v^2+2v+1 = 9$

$9-6u+u^2+v^2+2v+1 = 9[u^2+6u+9+v^2]$

$10-6u+u^2+v^2+2v-9u^2-54u-81-9v^2 = 0$

$-8u^2-60u-8v^2+2v-71 = 0$ $\times -1$

$8u^2+60u+8v^2-2v+71 = 0$ ①

$\left(u+\frac{15}{4}\right)^2+\left(v-\frac{1}{8}\right)^2 = \frac{333}{64}$ complete the square

$\therefore$ Radius is $\sqrt{\frac{333}{64}} = \frac{3\sqrt{37}}{8}$ ①, centre $= \left(-\frac{15}{4}, \frac{1}{8}\right)$ ①

Question 5 continued

Question 5 continued

(Total for Question 5 is 10 marks)

6.

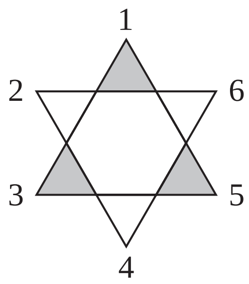


Figure 3

Figure 3 shows a plane shape made up of a regular hexagon with an equilateral triangle joined to each edge and with alternate equilateral triangles shaded.

The symmetries of this shape are the rotations and reflections of the plane that preserve the shape and its shading.

The symmetries of the shape can be represented by permutations of the six vertices labelled 1 to 6 in Figure 3. The set of these permutations with the operation of composition form a group, *G*.

(a) Describe geometrically the symmetry of the shape represented by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$$

(2)

(b) Write down, in similar two-line notation, the remaining elements of the group *G*.

(4)

(c) Explain why each of the following statements is false, making your reasoning clear.

(i) *G* has a subgroup of order 4

(ii) *G* is cyclic.

(2)

Diagram 1, on page 23, shows an unshaded shape with the same outline as the shape in Figure 3.

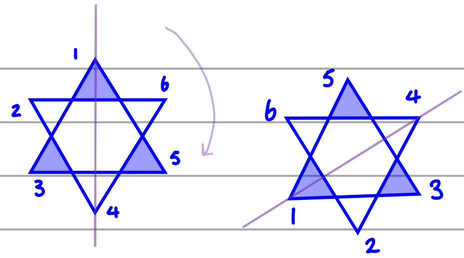
(d) Shade the shape in Diagram 1 in such a way that the group of symmetries of the resulting shaded shape is isomorphic to the cyclic group of order 6

(2)

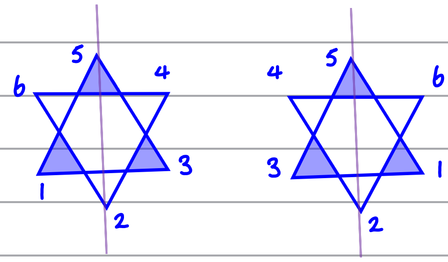
(a) A rotation 1 about the centre of the shape through an angle of 120° anticlockwise 1

Question 6 continued

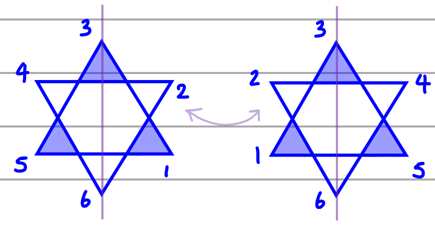
(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$ (1) e.g rotation 240° clockwise →



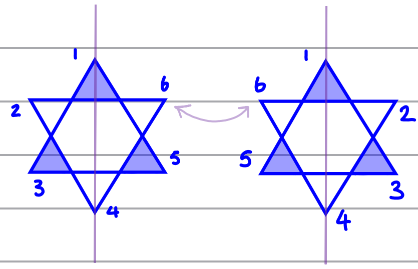
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$ (1) e.g reflection of rotation shown above →



$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$ e.g reflection of rotation given in (a) →



$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$ (1) e.g reflection in the vertical →



$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ (1)

(c) G has order 6 so cannot have a subgroup of order 4 by Lagrange's Theorem. (1)

order of subgroups must be a factor of the order of the group, and 4 is not a factor of 6.

There is no element of order 6 that generates the group. (1)

Question 6 continued

Question 6 continued

(d) Isomorphic = a one-to-one mapping between groups.

Must break all reflections, but keep all rotations

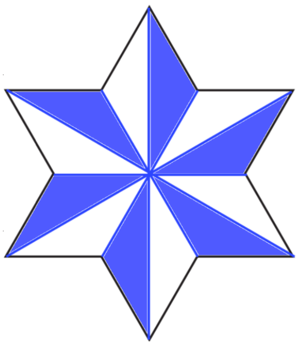


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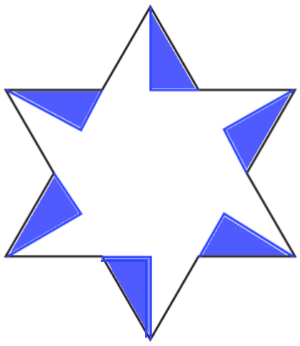
Diagram 1

Spare copy of Diagram 1

option 2.
(also 2)



Only use this diagram if you need to redraw your answer to part (d).

(Total for Question 6 is 10 marks)

7.

$$I_n = \int (4 - x^2)^{-n} \, dx \qquad n > 0$$

(a) Show that, for $n > 0$

$$I_{n+1} = \frac{x}{8n(4 - x^2)^n} + \frac{2n - 1}{8n} I_n$$

(5)

(b) Find I_2

(3)

(a) $I_n = \int 1 \times (4 - x^2)^{-n} \, dx$

Apply integration by parts : $\int uv' = uv - \int vu'$

$v' = 1 \implies v = x$

$u = (4 - x^2)^{-n} \implies u' = -2x \times -n \times (4 - x^2)^{-n-1}$

$I_n = x(4 - x^2)^{-n}$ 1 $+ \int 2x^2 n (4 - x^2)^{-n-1} \, dx$ 1

take out constant factor

$I_n = x(4 - x^2)^{-n} + 2n \int (4 - x^2 - 4)(4 - x^2)^{-n-1} \, dx$

$I_n = x(4 - x^2)^{-n} + 2n \int (4 - x^2)(4 - x^2)^{-n-1} - 4(4 - x^2)^{-(n+1)} \, dx$ 1

$I_n = x(4 - x^2)^{-n} + 2n \int (4 - x^2)^{-n} \, dx - 8n \int (4 - x^2)^{-(n+1)} \, dx$

$I_n = x(4 - x^2)^{-n} + 2n I_n - 8n I_{n+1}$ 1

$8n I_{n+1} = x(4 - x^2)^{-n} + 2n I_n - I_n$

$I_{n+1} = \frac{x(4 - x^2)^{-n}}{8n} + \frac{(2n - 1) I_n}{8n}$

$I_{n+1} = \frac{x}{8n(4 - x^2)^n} + \frac{(2n - 1) I_n}{8n}$ 1

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Question 7 continued

(b) $I_1 = \int \frac{1}{4-x^2}\,dx$ 1

in formula booklet

$I_1 = \frac{1}{2}\tanh^{-1}\left(\frac{x}{2}\right)$

$I_2 = \frac{x}{8(4-x^2)} + \frac{1}{8}I_1$ 1 ← using formula from (a), n = 1.

$I_2 = \frac{x}{8(4-x^2)} + \frac{1}{16}\tanh^{-1}\left(\frac{x}{2}\right) + c$ 1

Question 7 continued

Question 7 continued

(Total for Question 7 is 8 marks)

8. The four digit number $n = abcd$ satisfies the following properties:

(1) $n \equiv 3 \pmod 7$

(2) n is divisible by 9

(3) the first two digits have the same sum as the last two digits

(4) the digit b is smaller than any other digit

(5) the digit c is even

(a) Use property (1) to explain why $6a + 2b + 3c + d \equiv 3 \pmod 7$

(2)

(b) Use properties (2), (3) and (4) to show that $a + b = 9$

(4)

(c) Deduce that $c \equiv 5(a - 1) \pmod 7$

(2)

(d) Hence determine the number n, verifying that it is unique. You must make your reasoning clear.

(4)

(a) n can be written as $n = 1000a + 100b + 10c + d$ 1

$1000 \pmod 7 = 6 \pmod 7 \leftarrow 1000 = 147 \times 7 + 6$

$100 \pmod 7 = 2 \pmod 7 \leftarrow 100 = 14 \times 7 + 2$

$10 \pmod 7 = 3 \pmod 7 \leftarrow 10 = 1 \times 7 + 3$

$\therefore$ by reducing coefficients modulo 7,

$n \equiv 6a + 2b + 3c + d \pmod 7$ 1

(b) n is divisible by 9 $\Rightarrow a + b + c + d = 9k$

$a + d = b + c \Rightarrow a + d + a + d = 9k$

$2(a+d) = 9k$

$\therefore k$ is divisible by 2, so it is even.

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Question 8 continued

$a+b+c+d$ must be at least 3, and at most 35
(because b is smaller than all other numbers).

$1+0+1+1=3$ at least, $9+8+9+9=35$ at most

Since k must be even, the only possibility to keep
$3 \leq a+b+c+d \leq 35$ is $k=2$.

$\therefore\ a+b=9$

(c) Combining (a) and (b) gives.

$$3 \equiv 6a+2b+3c+d \pmod 7$$

$$3 \equiv 2(a+b)+4(a)+(c+d)+2c \pmod 7$$

$a+b=c+d=9$

$$3 \equiv 2(9)+4a+9+2c \pmod 7$$

$$3 \equiv 4a+2c+27 \pmod 7 \quad 1$$

$$2c \equiv -4a-24 \pmod 7$$

$$\equiv -4(a+6) \pmod 7$$ $\quad -4 \pmod 7 = 3 \pmod 7$

$$\equiv 3(a-1) \pmod 7$$ $\quad 6 \pmod 7 = -1 \pmod 7$

$\times 4$

$$8c \equiv 12(a-1) \pmod 7$$ $\quad 8 \pmod 7 = 1 \pmod 7$

$$c \equiv 5(a-1) \pmod 7 \quad 1$$ $\quad 12 \pmod 7 = 5 \pmod 7$

Question 8 continued

(d) $b > a$ and $a + b = 9$ so $5 \leq a \leq 9$. (1)

← if $a < 5$ then $b > a$

$a = 9 \Rightarrow c \equiv 5 \pmod 7 \Rightarrow c \equiv 5$ contradicts (5)

$a = 8 \Rightarrow c \equiv 0 \pmod 7 \Rightarrow c \equiv 0$ or 7 but $c \equiv 7$ contradicts (5), and

$b > 0$ because $a + b = 9 \rightarrow$ $b < c$ so can't be $c \equiv 0$.

$a = 7 \Rightarrow c \equiv 2 \pmod 7 \Rightarrow c \equiv 2$ but also $b = 2$ when $a = 7$, and $b < c$ so not possible. (1)

$a = 6 \Rightarrow c \equiv 4 \pmod 7 \Rightarrow c \equiv 4$, $d = 5$ while $a = 6$, $b = 3$ which

$a + b = c + d = 9 \longrightarrow$ works

$a = 5 \Rightarrow c \equiv 6 \pmod 7 \Rightarrow c \equiv 6$, $d = 3$ and $b = 4$, but then $d < b$ which contradicts (4). (1)

$\therefore n = 6345$ (1)

Question 8 continued

Question 8 continued

(Total for Question 8 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS